Hall Magnetohydrodynamics of weakly-ionized plasma

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We show that the Hall scale in a weakly ionized plasma depends on the fractional ionization of the medium and, Hall MHD description becomes important whenever the ion-neutral collision frequency is comparable to the ion-gyration frequency, or, the ion-neutral collisional mean free path is smaller than the ion gyro-radius. Wave properties of a weakly-ionized plasma also depends on the fractional ionization and plasma Hall parameters, and whistler mode is the most dominant mode in such a medium. Thus Hall MHD description will be important in astrophysical disks, dark molecular clouds, neutron star crusts, and, solar and planetary atmosphere.

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A wide range of laboratory and space phenomenon have been studied in the framework of Hall MHD. From magnetic flux expulsion in neutron star crusts [1] to angular momentum transport in weakly ionized protoplanetary disks [2], Hall effect provides the necessary physical mechanism for the plasma drift against the magnetic field. In the Earth's ionosphere, the Hall effect is important in determining the behaviour of electric currents generated by winds [3]. In fusion plasmas, the Hall effect can provide enhanced current drive via helicity injection into the ambient plasma [4].

Two approaches have been used to investigate the role of the Hall effect in plasma dynamics in laboratory [4], space [5, 6] and astrophysical [1, 2, 7] plasmas. In a highly ionized plasma, the Hall effect arises because of the difference in electron and ion inertia, which becomes important at frequencies comparable to or less than the ion cyclotron frequency, and therefore on a microscopic scale, the ion skin depth. In this case the Hall effect can be incorporated by explicitly including the ion-electron drift in the induction equation. In a weakly-ionised plasma, the Hall effect may instead arise because neutral collisions more easily decouple ions from the magnetic field than electrons. It's effects can be incorporated through a second-rank conductivity tensor appearing in a generalized Ohm's law [8, 9]. In this case, the Hall scale is macroscopic and can become comparable to the size of the system itself. The resulting dynamics are similar, but occur on very different scales due to the different mechanisms responsible for the underlying symmetry breaking in ion and electron dynamics. This has led to some confusion in the literature, where the fully-ionised estimate of the Hall length scale has been applied to partially ionized media to conclude that the Hall effect is irrelevant in circumstances when it is, in fact, crucial.

The purpose of this letter is to clarify the relationship between the fully-ionized and weakly ionised limits by developing a unified single-fluid framework for the dynamics of plasmas of arbitrary ionisation. Our treatment is of necessity approximate in the intermediate case, but has the correct limiting behaviour in the highly- or weakly-ionised limits. This allows us to to explore the change of scale in Hall effect in moving from fully to partially ionized plasmas and gain a deep physical understanding of the nature of the transition between the two ionisation regimes. Furthermore, this formulation is useful in gaining insight into the behaviour of plasmas that are in the intermediate regime, (e.g. near a tokamak wall or the surface of a white dwarf), where both collisions and plasma inertia becomes important. We also consider the wave modes in this formulation and demonstrate that the collisional whistler mode is of key importane in cold plasmas.

In the absence of finite ion Larmor radius effects, both electrons and ions are frozen in the magnetic field and single fluid MHD where ion carries the inertia and electron carries the current, is a good description. The Hall effect appears in a plasma when one of the plasma components is "unmagnetized". The spatial and temporal scale of this "unmagnetization" defines the Hall scale. The ´finite slip´ of the ions, which is the cause of this ´unmagnetization´ scales with the ion gyroradius ($\sim \sqrt{m_i}$, here m_i is the ion mass) over a time period $\leq$ ion gyration period. The cause of this ´slip´- ion inertia, is also responsible for introducing the helicity to the fluid. Therefore, Hall MHD of a fully ionized plasma introduces two disparate scale in the system - a ´kinetic´ scale due to the ion inertial effect and a macroscopic scale, that is typically of the order of the system size itself.

Cold space plasmas are generally collisional and weakly ionized. Their distinguishing feature is that the neutrals carry the inertia of the bulk fluid and the ionized component is operated upon by the Lorentz force. The electron-ion symmetry in a weakly-ionized plasma is as well broken by the ´finite ion slip´ against the electrons. However, since ion-neutral collision is the cause of this slip, the spatial and temporal scales of this symmetry breaking are quite different. Since the inertia of the fluid is carried by neutrals, collisional dynamics changes the scale of symmetry breaking, which becomes $\sim \sqrt{1/X_i}$, where $X_i = \rho_i/\rho_n$ (where $\rho_{i,n} = m_{i,n}\, n_{i,n}$ is the mass

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density of the ions and neutrals, $m_{i,n}$ represents mass of the ions and neutrals and $n_{i,n}$ is their number densities). In a weakly-ionized plasma, $X_i \ll 1$ and the ion inertial scale is often very large although it is not uncommon to find that even in a weakly ionized collisional space plasmas, the requirement on dynamical frequency ω of the system larger than ion-cyclotron frequency and ion-gyroradius larger than scale-length of interest is imposed to justify the validity of Hall description [5, 6]. We should expect that as the cause of symmetry breaking in a weakly-ionized plasma is due to ion-neutral collisions, the collision frequency must be larger than the ion-cyclotron frequency. This is a well known feature of collisional Hall MHD [3] although never explicitly derived using dynamical equations.

To illustrate these points we consider a partially ionised, magnetised, cold, quasineutral plasma consisting of ions, electrons and neutrals that are coupled by collisions. We start with the equations describing this three-component plasma and reduce then to a single-fluid description valid for arbitrary degrees of ionisation. Electron inertia is neglected, so that our treatment is not valid at frequencies higher than the electron cyclotron frequency. We then show how the scale below which the Hall effect becomes important increases from the ion gyroradius for a fully-ionised plasma to much larger scales for a weakly ionised plasma. We consider the Alfvénic modes in the plasma and show how the Hall effect introduces whistler type behaviour. Finally we discuss the implication for the role of the Hall effect in partially ionised plasmas in tokamaks, the ionosphere, the base of the solar photosphere, to protoplanetary discs, circumnuclear discs in active galactic nuclei and neutron stars.

We begin with the equations of continuity for each species:

$$\frac{\partial \rho_j}{\partial t} + \nabla\cdot(\rho_j\,\mathbf{v}_j) = 0\,, \tag{1}$$

where ρ_j is the mass density and $\mathbf{v}_j$ is the velocity of species j ($j = n, i, e$). The momentum equations for electrons, ions and neutrals are

$$0 = -e\,n_e\left(\mathbf{E} + \frac{\mathbf{v}_e\times\mathbf{B}}{c}\right) - \rho_e \sum_{j=i,n} \nu_{ej}\,(\mathbf{v}_e - \mathbf{v}_j) \tag{2}$$

$$\rho_i \frac{d\mathbf{v}_i}{dt} = e\,n_i\left(\mathbf{E} + \frac{\mathbf{v}_i\times\mathbf{B}}{c}\right) - \rho_i \sum_{j=e,n} \nu_{ij}\,(\mathbf{v}_i - \mathbf{v}_j) \tag{3}$$

$$\rho_n \frac{d\mathbf{v}_n}{dt} = \sum_{j=e,i} \rho_j\,\nu_{jn}\,(\mathbf{v}_j - \mathbf{v}_n) \tag{4}$$

The electron and ion momentum equations (2-3) contain terms on the right hand side for the Lorentz force and collisional momentum exchange, where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field, c is the speed of light and $\nu_{jk} \equiv \rho_k\,\gamma_{jk} = \rho_k\, <\sigma\,v>_j / (m_k + m_j)$ is the collision frequency of species j with species k.

To derive a single-fluid description, we define the mass density of the bulk fluid $\rho = \rho_e + \rho_i + \rho_n \approx \rho_i + \rho_n$, and the bulk velocity

$$\mathbf{v} = (\rho_n\mathbf{v}_n + \rho_i\mathbf{v}_i)/\rho \tag{5}$$

The continuity equation is simply derived by summing (1) over j

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{v}) = 0. \tag{6}$$

To derive the momentum equation for the total fluid, use eq (1) for the ions and neutrals and the sum of the momentum equations (2), (3) and (4) to obtain

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot\left(\rho\mathbf{v}\mathbf{v} + \frac{\rho_i\rho_n}{\rho}\mathbf{v_d}\mathbf{v_d}\right) = \frac{\mathbf{J}\times\mathbf{B}}{c} \tag{7}$$

where $\mathbf{v_d} = \mathbf{v_i} - \mathbf{v_n}$ is the ion-neutral drift velocity. The second term on the LHS is simply the divergence of $\rho_i\mathbf{v}_i\mathbf{v}_i + \rho_n\mathbf{v}_n\mathbf{v}_n$. Noting that $\rho_i\rho_n/\rho \le \frac{1}{4}\rho$, we may neglect the $\mathbf{v_d}\mathbf{v_d}$ term if gradients in $\mathbf{v_d}$ are small compared to those in $\mathbf{v}$. We shall derive a self-consistency condition for this below, meanwhile with this assumption and using (6) we recover the usual single, cold fluid momentum equation:

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{\mathbf{J}\times\mathbf{B}}{c\,\rho}\,. \tag{8}$$

where $\mathbf{J} = e\,n_e\,(\mathbf{v}_i - \mathbf{v}_e)$ is the current density.

To estimate $\mathbf{v_d}$, we first note that the equivalents of (8) for the ionised and neutral components seperately are

$$\frac{\partial\mathbf{v}_n}{\partial t} + (\mathbf{v}_n\cdot\nabla)\mathbf{v}_n = \gamma\rho_i\mathbf{v_d} \tag{9}$$

and

$$\frac{\partial\mathbf{v}_i}{\partial t} + (\mathbf{v}_i\cdot\nabla)\mathbf{v}_i = \frac{\mathbf{J}\times\mathbf{B}}{c\,\rho_i} - \gamma\rho_n\mathbf{v_d}\,. \tag{10}$$

In the limit $\rho_i \ll \rho_n$ $\mathbf{v}_n \to \mathbf{v}$ and therefore

$$\mathbf{v_d} \approx \frac{\mathbf{J}\times\mathbf{B}}{c\gamma\rho\rho_\mathbf{i}}\,. \tag{11}$$

to high accuracy. This is the strong coupling approximation commonly adopted for weakly-ionised gas [10], for which ρ in (11) is replaced by ρ_n. More generally, provided that $|\mathbf{v_d}| \ll |\mathbf{v}|$ – in other words, that the single fluid treatment is valid – the right hand sides of eqs (8)–(10) should be very similar, and we again deduce that $\mathbf{v_d}$ still holds. To use this expression for $\mathbf{v_d}$ we must demand $v_d \lesssim v_A \equiv B/\sqrt{4\,\pi\,\rho}$, which in turn (again using eq [11]) implies that

$$\omega \lesssim \nu_{ni} \tag{12}$$

The expression (11) for $\mathbf{v_d}$ implies that the $\mathbf{v_d}\mathbf{v_d}$ term in (7) can be neglected for dynamical frequencies satisfying

$$\omega \lesssim \frac{\rho}{\sqrt{\rho_i\rho_n}}\,\nu_{ni}\,. \tag{13}$$

At higher frequencies, the single-fluid approximation (8) breaks down. Note that much higher frequencies can be tolerated in the weakly ionized ($\omega/\nu_{ni} \leq 1/\sqrt{X_i}$) or almost completely ionized ($\omega/\nu_{ni} \leq \sqrt{X_i}$) limits.

Using (11) to compute $\mathbf{E}$ in terms of ion-electron and ion-neutral drifts, and expressing both in terms of $\mathbf{J}$, we find that the induction equation in the bulk frame can be written as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\left[(\mathbf{v}\times\mathbf{B}) + D\frac{(\mathbf{J}\times\mathbf{B})\times\mathbf{B}}{c\,\rho_i\,\nu_{in}} - \frac{\mathbf{J}\times\mathbf{B}}{e\,n_e} - \eta\,\mathbf{J}\right] \quad (14)$$

where $D = \rho_n/\rho$, $\eta = n_e m_e\left(\nu_{en} + \nu_{ei}\right)/e^2$ is the Ohmic resistivity, and we have neglected terms of order ν_{en}/ν_{in}.

Equations (6), (8), and (14) describe the dynamics of a partially or fully ionized plasma. For example, when the plasma is fully ionized, (i.e. $D \to 0$), $\mathbf{v} = \mathbf{v}_i$ and (6), (8), and, (14) reduces to fully ionized Hall MHD description. In the other extreme limit $D \to 1$, the equations reduce to those describing weakly ionized MHD [2, 7]. There is an additional missing ingredient: a model for the dependence of n_e (or ρ_i) on ρ. This is not needed for our purposes here, ie. estimating the scales on which the Hall term is relevant, and exploring the propagation of non-compressive modes along the magnetic field.

Adopting the standard estimates, $v \sim v_A$, $\omega \sim v_A/L$ where v_A is the Alfven speed in the combined fluid and L is the characteristic scale of a disturbance, the Hall term dominates the inductive term in (14) for frequencies higher than ω_H, where

$$\omega_H = \frac{\rho_i}{\rho}\frac{eB}{m_i c}\,. \quad (15)$$

with corresponding lengthscales below $L_H = v_A/\omega_H$. For a fully ionised plasma, we recover the standard criterion, that the Hall effect comes into play for frequencies in excess of the ion cyclotron frequency ($\omega_{ci} = e\,B/m_i\,c$), corresponding to lengthscales below the ion gyroradius. These frequencies are usually much higher and the lengthscales are much shorter than those of interest and so the Hall effect is generally considered to be unimportant. However, for a partially ionized plasma, the critical frequency is lower by a factor of ρ_i/ρ, and the lengthscales are longer by the same factor. This ratio can be very low, thus in a weakly ionised plasma, ω_H and L_H can become comparable to the dynamical frequency and/or scale of the system under consideration and the Hall effect plays a key role in the evolution of the entire system.

The physical interpretation for the scaling (15) is clear: the coupling of ions to the neutrals through collisions that is required for the single-fluid momentum equation to be valid gives each ion an effective mass that ρ/ρ_i times m_i. Their effective gyrofrequency is reduced by the same factor.

Implicit in this is the requirement that collisions are able to do the job, i.e. $\omega \lesssim \sqrt{X_i}\,\nu_{in}/D$. Writing ω_H in terms of collision parameters $\beta_i = \omega_{ci}/\nu_{in}$, $\omega_H = (\rho_i/\rho)\,\beta_i\,\nu_{in}$, we see that Hall description is valid if $\nu_{in} \gtrsim \omega \gtrsim \omega_H$, i.e.

$$\left(\frac{1+X_i}{X_i^{1.25}}\right)^2 > \beta_i\,. \quad (16)$$

Although in a WIP, when $X_i \ll 1$, the Hall effect can operate for a wide range of β_i value due to condition (16), it ceases to be important in comparision with the ambipolar diffusion once $\beta \gtrsim 1$, as the ratio between the ambipolar and the Hall term is $\sim \beta_i$. When $X_i \lesssim 1$, then Hall effect can operate in a much narrow β_i range since $\sqrt{X_i}\,\beta_i \ll 1$. The scale of the Hall MHD becomes a function of fractional ionization and ion-Hall parameter

$$L_H = \frac{\rho}{\rho_i}\left(\frac{v_A}{\nu_{in}}\right)\beta_i^{-1}\,. \quad (17)$$

Thus in a partially ionized, collision dominated plasma, where $\beta_i \lesssim 1$, L_H can become very large.

Now we investigate the wave properties of the medium in the presence of only Hall term in the induction equation (14) and explore its properties in various fractional ionization limit. We assume a homogeneous background with no flow ($\mathbf{v}_0 = 0$) and look for transverse fluctuations propagating along the magnetic field (i.e. $\delta\mathbf{B} \perp \mathbf{B}$ and $\hat{\mathbf{k}}\cdot\hat{\mathbf{B}} = 1$) of the form $\exp\left(i\,\omega\,t + i\,\mathbf{k}\cdot\mathbf{x}\right)$ where ω is the angular frequency and $\mathbf{k}$ is the wave vector. With these assumptions we obtain the dispersion relation

$$\omega^2 = \omega_A^2 \pm \left(\frac{\omega_A^2}{\omega_H}\right)\omega. \quad (18)$$

Here $\omega_A = k\,v_A$ is the Alfvén frequency. When $\rho_i \approx \rho_n$, reduces to the equation (18) of [11] except for a numerical factor 1/2 owing to the presence of the neutrals. When $\omega_A \ll \omega_{ci}$ we recover shear Alfvén waves $\omega^2 = \omega_A^2$, and when $\omega_{ci} \ll \omega_A$, electrostaic ion-cyclotron $\omega^2 = \omega_{ci}^2$, and, electromagnetic, whistler waves $\omega^2 = (\omega_A^2/\omega_{ci})^2$ are recovered.

In the present formulation, when $\omega_H \ll \omega_A$, i.e. $L \ll L_H$, the dispersion relation (18) gives electrostatic ion-cyclotron waves, $\omega^2 = \omega_H^2$ and electromagnetic whistler waves

$$\omega^2 = \left(\frac{\omega_A^2}{\omega_H}\right)^2\,. \quad (19)$$

However, unlike highly ionized medium, in a weakly ionized medium, excitation of the ion-cyclotron mode is difficult since $\omega_H \approx 0$. When $\omega_A \ll \omega_H$, i.e. $L_H \ll L$, the dominant wave is the polarized Alfvén wave, $\omega^2 = \omega_A^2$. In a WIP, X_i and β_i can become very small [12–14], and, wavelength of the Alfvén mode can become larger than the system size.

The propagation of waves in a weakly ionized medium have important applications. For example, the observation of the partially ionized lower boundary of the Earth's ionosphere ($\sim 70 - 140\,$kms), consisting of E

TABLE I: The typical parameters of the weakly ionized medium is given in the table.

	$\frac{n_n}{\mathrm{cm}^3}$	X_e	$\frac{B}{\mathrm{G}}$	$\frac{v_A}{\mathrm{km/s}}$	ω_H/s	β_i	$\frac{L_H}{\mathrm{km}}$
Earth[a]	10^{12}	10^{-8}	0.3	$\lesssim 1$	10^{-6}	10^{-2}	10^{6}
Sun[b]	10^{17}	10^{-4}	10^{3}	10	10^{2}	10^{-3}	10
PPD[c]	10^{15}	10^{-12}	1	10^{-2}	10^{-8}	10^{-4}	10^{9}
NS[d]	10^{34}	10^{-5}	10^{12}	22	10^{11}	10^{-9}	10^{-10}

[a]Earth's E-region $\sim 80-150\,$km, from Ref. [12]. For calculation purpose, ions are assumed to consist only of ionized Oxygen atoms.
[b]Sun's Photosphere $< 500\,$km, from Ref. [13]. An average magnetic field ~ 1 G is assumed. Ions are mainly metallic in the lower photosphere and we have assumed equal ion and neutral masses.
[c]Protoplanetary disks parameters at 1 au are taken from Ref. [14].
[d]For neutron star, we have assumed a fractional ionization 10^{-5} from the fact that ions densities vary from $\sim 10^{6}$ g/cm^3 to 10^{11} g/cm^3 in the neutron star crust [16]. The ion mass $m^* = 0.8\,m_p$.

and D regions reveal the permanent presence of ULF waves. In the E-region of the ionosphere, such waves have slow and fast components with phase velocities between $1-100\,\mathrm{m\,s^{-1}}$ and $2-20\,\mathrm{km\,s^{-1}}$ and frequencies between $(10^{-1}-10^{-4})\,\mathrm{s^{-1}}$ and $(10^{-4}-10^{-6})\,\mathrm{s^{-1}}$ respectively with wavelength $\gtrsim 10^3\,$km and a period of variation ranging between few days to tens of days [15]. The ULF waves have been identified as Alfvén and whistler waves [12]. The Hall scale from table-I suggests that only whistlers with wavelengths $\sim 10^6$ km can be excited in the lower ionosphere.

Wave heating of the solar corona is thought to be due to the Alfvén wave that have emerged in the lower photosphere, possibly excited by the foot point motion of the magnetic field. The present investigation suggests that whistler waves with wavelength $\gtrsim 10\,$km can be excited in the lower photosphere. A frequency power spectrum for horizontal photospheric motions [17] shows that waves of smaller frequencies $10^{-5}-0.1Hz$ can be observed at a few solar radii. Thus, it should be possible to observe 1 Hz waves which would confirm the existence of the whistler in the solar photosphere.

In PPDs, the Hall effect will be important for very low frequencies $\omega \geq 10^{-8}\,$Hz. The value of L_H ratio (table-I) suggests that whistlers will be the dominant mode in the disk. In neutron stars too, very small wavelength whistlers will be excited in the crust fluid.

To summarize, (i) the Hall criteria of two fluid MHD is not suitable in the Hall description of a WIP. In a weakly ionized medium collisions can decouple electron and ion motions over sizeable part of the system. Hall MHD is valid for a WIP if $\omega^{-1} \lesssim \omega_H^{-1}$ and $L_H \gtrsim L$. In space plasmas, if the medium is weakly ionized, i.e. $X_i \to 0$ and ω_H is smaller than the ion-cyclotron frequency by a factor $X_i/(1+X_i)$. Therefore, the requirement that the dynamical frequency of the system is larger than the modified ion-cyclotron frequency is easy to satisfy. Furthermore, the Hall scale L_H is inversely proportional to the fractional ionization. This explains why the Hall MHD is so important to weakly ionized space environments.

(ii) The temporal scale of Hall MHD in a WIP is related to the ion-neutral collision frequency in addition to the dynamical frequency.

(iii) The collision induced Hall MHD excites long wavelength whistler mode. In a WIP whistler appears to be the only locally excitable mode.

(iv) Whistler apperas to be the most dominant mode in the Earth's lower ionosphere, solar photosphere, planetary disks, and, neutron stars.

To conclude, Hall MHD in a WIP operates on very large scale and is crucial to the dynamics of the medium. The propagation of the whistler mode in the weakly ionized medium is dependent upon the fractional ionization. Therefore, in a WIP long wavelength whistler can be easily excited.

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